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Studying random differential equations as a tool for turbulent diffusion

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A modification of Kraichnan's direct interaction approximation (DIA) [R. H. Kraichnan, J. Math. Phys. **2**, 124 (1961)] for random linear partial differential equations is proposed. The approximation is tested on the specific example of the turbulent advection of a scalar quantity by a random velocity field. It is shown to account for the sweeping more correctly than the DIA. As a result, it is valid for all times and it is able to describe nonstandard diffusive processes (i.e., superdiffusive and non-Gaussian) for which the DIA is not valid. [S1063-651X(98)50111-0]

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Stochastic modeling plays an important role in many branches of physics. Generally, it amounts to studying the following linear partial differential equation for the scalar quantity $\rho(\boldsymbol{r},t)$ evolving on the d-dimensional phase space $\{\boldsymbol{r}\}$:

$$\frac{\partial}{\partial t}\rho(t)=l\left(\boldsymbol{r},\frac{\partial}{\partial \boldsymbol{r}},t\right)\rho(t), \tag{1}$$

where the operator $l(\boldsymbol{r},\partial/\partial\boldsymbol{r},t)$ is random with specified statistics. With appropriate identification, Eq. (1) may model a wide variety of physical situations. Just to mention one of particular interest, Eq. (1) may be the Liouville or the Fokker-Planck equation associated with a set of random *nonlinear* ordinary differential equations, and the method presented below may thus serve studying these equations as well. The complete solution of Eq. (1) amounts to determining the statistics of $\rho(t)$. Here, we shall determine only the mean value $\langle\rho(t)\rangle$, which is of special interest since it fixes the one-point statistics of any quantity evolving on the phase space $\{\boldsymbol{r}\}$. Despite the linear character of Eq. (1), averaging of this equation leads to the highly nontrivial closure problem of determining $\langle l(t)\rho(t)\rangle$. One of the most powerful approximations for the solution of this problem is Kraichnan's direct interaction approximation (DIA) [1]. However, the DIA has the defect that it misrepresents the sweeping; that is, the effect of the large scales of $l(t)$ on the small scales of $\rho(t)$. In particular, the DIA is not valid for short times, and gives reliable information on the asymptotic dynamics of $\langle\rho(t)\rangle$ only if the sweeping becomes negligible as $t\to\infty$, which need not be the case. In this Rapid Communication, we propose an approximation, referred to as the modified direct interaction approximation (MDIA) which accounts for the sweeping more correctly than the DIA. In particular, the MDIA is valid for all times and is asymptotically equivalent to the DIA only if the sweeping is negligible as $t\to\infty$. Since the MDIA is worked out in the Eulerian frame, it is an interesting alternative to the complicated Lagrangian modifications of the DIA [3].

We shall present the MDIA for the general case, then apply it on the problem of the passive advection of a scalar quantity by a three dimensional incompressible random velocity field, a problem that arises in many contexts (mass, charge, and heat transport in turbulent fluid, in porous media, etc.) Then Eq. (1) specializes to

$$\frac{\partial}{\partial t}\rho(t)=-\boldsymbol{v}(\boldsymbol{r},t)\cdot\boldsymbol{\nabla}\rho(t)+D_0\Delta\rho(t), \tag{2}$$

where D_0 is the molecular diffusion coefficient and the velocity $\boldsymbol{v}(\boldsymbol{r},t)$ is taken to be a Gaussian random process, statistically isotropic (hence zero-mean and homogeneous) and stationary. Then, the statistics of $\boldsymbol{v}(\boldsymbol{r},t)$ are fully specified by the scalar covariance ($r^2=\boldsymbol{r}\cdot\boldsymbol{r}$),

$$\langle\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{r}',t+t')\cdot\boldsymbol{v}(\boldsymbol{r}',t')\rangle\equiv 2\int_0^\infty dk\,\frac{\sin(kr)}{kr}E(k,t).$$

The quantity $E(k,t)$ will be referred to as the energy spectrum. By definition, it is normalized as

$$\int_0^\infty dk E(k,0)=\frac{3}{2}v_\star^2, \tag{3}$$

where $v_\star$ is the root-mean-square velocity in any direction, and, using $E(k,t)$, we shall measure characteristic length-scale $l_\star$ and time-scale $t_\star$ of the velocity from

$$l_\star^2=\int_0^\infty dk\,\frac{E(k,0)}{v_\star^2k^2},\quad t_\star=\int_0^\infty dt\int_0^\infty dk\,\frac{E(k,t)}{v_\star^2}. \tag{4}$$

Note that we do *not* require that these integrals be finite; we may have $l_\star=\infty$, meaning that much of the energy is concentrated in the large scales of $\boldsymbol{v}(\boldsymbol{r},t)$, or $t_\star=\infty$, meaning that $\boldsymbol{v}(\boldsymbol{r},t)$ has no effective decorrelation with time. As applied to Eq. (2), we will show that the MDIA is valid for all times since it accounts correctly for the sweeping effects. Furthermore, according to the MDIA, the asymptotic dynamics of $\langle\rho(t)\rangle$ depends dramatically on $l_\star$ and $t_\star$. Specifically, provided only that either $l_\star$ or $t_\star$ is finite, the MDIA equation for $\langle\rho(t)\rangle$ reduces, as $t\to\infty$, to a diffusion equation with an effective diffusion coefficient $D_\star$ which, in the limit $D_\star\gg D_0$, depends on some combination of $v_\star$, $l_\star$ and $t_\star$ only [see Eq. (22)]. However, if both $l_\star$ and $t_\star$ are infinite, the MDIA equation never reduces to a diffusion equation;

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the asymptotic dynamic is superdiffusive and non-Gaussian, and we will be able to quantify both these features [see Eqs. (23) and (24)]. In particular, since $l_\star=\infty$ in the superdiffusive range, the large scale effects are essential, the sweeping is never negligible and, hence, the DIA cannot be used.

We now present the MDIA for Eq. (1). To this end, we introduce first the evolution operator $h(t|t')$ defined as

$$\rho(\boldsymbol{r},t)=h(t|t_0)\rho_0(\boldsymbol{r}). \tag{5}$$

Applying $h(t|t')$ to the distribution $\delta(\boldsymbol{r}-\boldsymbol{r}')$ defines the Green function of Eq. (1): $g(\boldsymbol{r},t|\boldsymbol{r}',t')\equiv h(t|t')\delta(\boldsymbol{r}-\boldsymbol{r}')$. The operator $h(t|t')$ satisfies Eq. (1), which we write as

$$\frac{\partial}{\partial t}h(t|t')=[L(t)+\lambda\tilde{l}(t)]h(t|t'), \tag{6}$$

with the initial condition $h(t|t)=I$, I being the identity operator. In Eq. (6), λ is an ordering parameter introduced for convenience and we have decomposed $l(t)$ into an averaged and a purely random part: $L(t)=\langle l(t)\rangle$ and $\lambda\tilde{l}(t)=l(t)-L(t)$. We shall derive from Eq. (6) an equation for $H(t|t')\equiv\langle h(t|t')\rangle$; from Eq. (5), we have $\langle\rho(\boldsymbol{r},t)\rangle=H(t|t_0)\rho_0(\boldsymbol{r})$ [assuming that $\rho_0(\boldsymbol{r})$ is deterministic] and $H(t|t')$ is related to the averaged Green function $G\equiv\langle g\rangle$ of Eq. (1) as $G(\boldsymbol{r},t|\boldsymbol{r}',t')\equiv H(t|t')\delta(\boldsymbol{r}-\boldsymbol{r}')$. Introducing the operator $H_0(t|t')$ satisfying

$$\frac{\partial}{\partial t}H_0(t|t')=L(t)H_0(t|t'),\quad H_0(t|t)=I, \tag{7}$$

it follows that Eq. (6) may be rewritten as

$$h(t|t')=H_0(t|t')+\lambda\int_{t'}^{t}dsH_0(t|s)\tilde{l}(s)h(s|t'). \tag{8}$$

Upon iterating and averaging, Eq. (8) leads to the following infinite perturbation series expansion for $H(t|t')$:

$$H(t|t')=H_0(t|t')+\lambda^2H_2(t|t')+\lambda^3H_3(t|t')+\cdots, \tag{9}$$

where $H_0(t|t')$ was defined in Eq. (7) and, e.g.,

$$H_2(t|t')=\int_{t'}^{t}ds\int_{t'}^{s}ds'H_0(s|s')\times\langle\tilde{l}(s)H_0(s|s')\tilde{l}(s')\rangle H_0(s'|t'). \tag{10}$$

Truncation of Eq. (9) at any order yields an approximation for $H(t|t')$ only valid on a finite time interval. Improvement is obtained by partial resummation (or renormalization) of the series (9). One way to proceed is as follows. Consider the following two equations for $H(t|t')$:

$$\frac{\partial}{\partial t}H(t|t')=L(t)H(t|t')+\lambda^2\int_{t'}^{t}dsQ(t,s)H(s|t'), \tag{11a}$$

$$\frac{\partial}{\partial t}H(t|t')=L(t)H(t|t')+\lambda^2P(t,t')H(t|t'). \tag{11b}$$

With appropriate definitions for the operator $Q(t,t')$ and $P(t,t')$ both equations are exact even if formal, since exact evaluation of $Q(t,t')$ and $P(t,t')$ is restricted to some simple cases. Yet, it may be proven that renormalization of the series (9) may be achieved by appropriate approximations for $Q(t,t')$ or $P(t,t')$. Kraichnan's DIA is one such approximation for $Q(t,t')$ [see Eq. (15a)]. In this Rapid Communication, we shall rather work with Eq. (11b) and the MDIA will be obtained as some approximation for $P(t,t')$ [see Eq. (15b)]. The expansions of $Q(t,t')$ and $P(t,t')$ may be easily found from that of $H(t|t')$ and Eq. (11)

$$\lambda^2Q(t,t')=\lambda^2Q_2(t,t')+\lambda^3Q_3(t,t')+\cdots, \tag{12a}$$

$$\lambda^2P(t,t')=\lambda^2P_2(t,t')+\lambda^3P_3(t,t')+\cdots, \tag{12b}$$

where up to order λ^2

$$Q_2(t,t')=\langle\tilde{l}(t)H_0(t|t')\tilde{l}(t')\rangle, \tag{13a}$$

$$P_2(t,t')=\int_{t'}^{t}ds\langle\tilde{l}(t)H_0(t|s)\tilde{l}(s)\rangle H_0^{-1}(t|s). \tag{13b}$$

In Eq. (13b), the operator $H_0^{-1}(t|t')$ is the inverse of $H_0(t|t')$ [i.e., $H_0(t|t')H_0^{-1}(t|t')=H_0^{-1}(t|t')H_0(t|t')=I$]. Expansion (12b) is the so-called time-ordered cumulant expansion [2]. Truncation of Eqs. (12a) and (12b) at order λ^2 leads respectively to the quasinormal approximation (QNA) and the quasilinear approximation (QLA). Roughly, both these approximations are valid for all times if $H(t|t')$ evolves on a time scale that is much longer than the time scale associated with $\tilde{l}(t)$. For approximations based on Eq. (11a), it is well-known that reworking the series for $Q(t,t')$ as a series in $H(t|t')$ rather than $H_0(t|t')$ leads to improved approximations, since it achieves a further resummation of the original series (9). We claim that a similar improvement is obtained for approximations based on Eq. (11b) by reworking the series for $P(t,t')$ as a series in $H(t|t')$ rather than $H_0(t|t')$, since it also achieves a further resummation of the series (9). This gives

$$\lambda^2Q(t,t')=\lambda^2\overline{Q}_2(t,t')+\lambda^3\overline{Q}_3(t,t')+\cdots, \tag{14a}$$

$$\lambda^2P(t,t')=\lambda^2\overline{P}_2(t,t')+\lambda^3\overline{P}_3(t,t')+\cdots, \tag{14b}$$

where up to order λ^2

$$\overline{Q}_2(t,t')=\langle\tilde{l}(t)H(t|t')\tilde{l}(t')\rangle, \tag{15a}$$

$$\overline{P}_2(t,t')=\int_{t'}^{t}ds\langle\tilde{l}(t)H(t|s)\tilde{l}(s)\rangle H^{-1}(t|s), \tag{15b}$$

In Eq. (15b), the operator $H^{-1}(t|t')$ is the inverse of $H(t|t')$ [i.e., $H(t|t')H^{-1}(t|t')=H^{-1}(t|t')H(t|t')=I$]. Truncation of Eq. (14a) at order λ^2 is the DIA; truncation of Eq. (14b) at order λ^2 will be referred to as the MDIA. Explicit equations for $\langle\rho(\boldsymbol{r},t)\rangle$ or $G(\boldsymbol{r},t|\boldsymbol{r}',t')$ may be obtained by applying Eq. (11) on $\rho_0(\boldsymbol{r})$ or $\delta(\boldsymbol{r}-\boldsymbol{r}')$, respectively.

The MDIA deserves the following comments:

(i) *The MDIA is a well-defined approximation even though the averaged dynamics is usually irreversible and the inverse evolution operator $H^{-1}(t|t')$ appears in Eq. (15b).* In particular, the reversibility of $H(t|t')$, which we assume here, does not require the averaged dynamics to be reversible. Of course, for irreversible dynamics, the inverse of the averaged Green function, which would be defined as $G^{-1}(\boldsymbol{r},t|\boldsymbol{r}',t')\equiv H^{-1}(t|t')\delta(\boldsymbol{r}-\boldsymbol{r}')$, does not exist, meaning simply that $\delta(\boldsymbol{r}-\boldsymbol{r}')$ does not belong to the domain of definition of the operator $H^{-1}(t|t')$. Yet, this leads to no difficulty for the MDIA because $\delta(\boldsymbol{r}-\boldsymbol{r}')$ belongs to the domain of definition of the operator $H^{-1}(t|s)H(t|t')$, since $t'\leqslant s\leqslant t$.

(ii) *The MDIA and the DIA are asymptotically equivalent if and only if the averaged dynamics is approximately Markovian as* $t\to\infty$. Non-Markovianity is $H(t|t')\neq H(t|t'')H(t''|t')$ $(t\geqslant t''\geqslant t')$ and is implied by both the DIA and the MDIA [unless $l(t)$ is a white-noise process and even though $h(t|t')=h(t|t'')h(t''|t')$ by definition]. Thus, the term $H^{-1}(t|s)H(t|t')$ in the MDIA equation cannot be approximated by the term $H(s|t')$ in the DIA equation unless the averaged dynamics becomes Markovian as $t\to\infty$. Below, we relate the relaxation to Markovianity to the absence of sweeping as $t\to\infty$.

(iii) *In contrast with the DIA, the MDIA is valid for the short times.* Direct calculation shows that, in contrast with the DIA, the MDIA is exact for a Gaussian $l(t)$ in Eq. (1) as long as the following local approximation is valid: $l(\boldsymbol{r},\partial/\partial\boldsymbol{r},t)\approx l(\boldsymbol{r}',\partial/\partial\boldsymbol{r},t)$, where $\boldsymbol{r}'$ is some fixed position in phase space. Since initially $g(\boldsymbol{r},t|\boldsymbol{r}',t)=\delta(\boldsymbol{r}-\boldsymbol{r}')$, this local approximation is valid on some time interval, and the MDIA is indeed valid for the short times.

(iv) *The MDIA accounts for the sweeping more correctly than the DIA.* First, it results from point (iii) that, in contrast with the DIA, the MDIA is exact for pure sweeping; that is, if $l(\boldsymbol{r},\partial/\partial\boldsymbol{r},t)\equiv l(\partial/\partial\boldsymbol{r},t)$, Gaussian. Second, since the MDIA is local in time (and not convolutive, like the DIA), it appears to account more correctly than the DIA for the average effect of a so-called random Galilean transformation (RGT) [3]; that is, $l(\boldsymbol{r},\partial/\partial\boldsymbol{r},t)\to l(\boldsymbol{r}-\boldsymbol{v}t,\partial/\partial\boldsymbol{r},t)-\boldsymbol{v}\cdot\partial/\partial\boldsymbol{r}$, where the constant $\boldsymbol{v}$ is some Gaussian random variable statistically independent on $l(t)$. Note however that since the MDIA is worked out in an Eulerian and not a Lagrangian frame, it cannot account exactly for a RGT.

As an illustration, we now apply the MDIA on Eq. (2), i.e., on the specific problem of the advection a scalar quantity by a random velocity field (for the predictions of the DIA, see, e.g., Ref. [4]). Owing to the statistical isotropy and stationarity of the velocity field $\boldsymbol{v}(\boldsymbol{r},t)$, it follows that $H(t|t')=H(t-t')$ and $H(t)\delta(\boldsymbol{r}-\boldsymbol{r}')=G(|\boldsymbol{r}-\boldsymbol{r}'|,t)$. Here $G(r,t)$ may be interpreted as the probability density that a particle randomly advected by $\boldsymbol{v}(\boldsymbol{r},t)$ and subject to molecular diffusion is in $\boldsymbol{r}+\boldsymbol{r}'$ at time $t+t'$ if it was in $\boldsymbol{r}'$ at time t'. Also, the average of the solution of Eq. (2) is $\langle\rho(\boldsymbol{r},t)\rangle=\int d\boldsymbol{r}'G(|\boldsymbol{r}-\boldsymbol{r}'|,t-t_0)\rho_0(\boldsymbol{r}')$. Introducing the Fourier representation

$$\hat{G}(k,t)=4\pi\int_0^\infty dr r^2\frac{\sin(kr)}{kr}G(r,t),$$

the MDIA leads after some manipulations to

$$\frac{\partial}{\partial t}\hat{G}(k,t)=-k^2[D_0+\eta(k,t)]\hat{G}(k,t), \tag{16}$$

with the initial conditions $\hat{G}(k,0)=1$ and where

$$\eta(k,t)=\int_0^t ds\int_0^\infty dppK(k,p,s)\,\frac{\hat{G}(p,s)}{\hat{G}(k,s)}, \tag{17}$$

$$K(k,p,t)=\frac{2}{\pi}\int_{|k-p|}^{k+p}dq\,\frac{\sin^2(k,q)}{kq}E(q,t), \tag{18}$$

in which sin(k,q) is the sine of the interior angle opposite to p in a triangle of sides k, p and q.

As a first straightforward consequence of Eq. (16), we note that for a pure sweeping flow, $\boldsymbol{v}(\boldsymbol{r},t)\equiv\boldsymbol{v}(t)$, for which $E(k,t)=\frac{1}{2}\delta(k)\langle\boldsymbol{v}(t)\cdot\boldsymbol{v}(0)\rangle$, the MDIA is exact since it leads to the Gaussian

$$\hat{G}(k,t)=e^{-k^2D_0t-k^2w(t)}, \tag{19}$$

where $w(t)=\frac{2}{3}\int_0^t ds(t-s)\langle\boldsymbol{v}(s)\cdot\boldsymbol{v}(0)\rangle$. In contrast, the DIA equations are wrong in this case: they can be solved analytically in the time representation only for a time-independent sweeping flow, $\boldsymbol{v}(\boldsymbol{r},t)\equiv\boldsymbol{v}$, where they lead to $\hat{G}(k,t)=J_1(2kv_\star t)/(kv_\star t)$ [here, $J_1(x)$ is a modified Bessel function] [4], while the exact result (19) is in this case $\hat{G}(k,t)=\mathrm{e}^{-k^2D_0t-(1/2)k^2v_\star^2t^2}$. Note also that, if t is small enough, $\hat{G}(k,t)$ is mostly flat compared to the spectrum $E(k,t)$, and the flow can be approximated by a pure sweeping flow. Consequently, in contrast with the DIA, the MDIA is valid for the short times.

To further argue that the MDIA accounts correctly for the sweeping, we consider the effect on Eq. (16) of the RGT $\boldsymbol{v}(\boldsymbol{r},t)\to\overline{\boldsymbol{v}}(\boldsymbol{r},t)\equiv\boldsymbol{v}(\boldsymbol{r}-\boldsymbol{u}t,t)+\boldsymbol{u}$, where the constant velocity $\boldsymbol{u}$ is a statistically isotropic Gaussian random variable, statistically independent on $\boldsymbol{v}(\boldsymbol{r},t)$ and whose mean square value in any direction is $u_\star^2$. The RGT adds a term $-k^2u_\star^2\hat{G}(k,t)$ on the right-hand-side of Eq. (16), and from Eq. (17) it follows that the effect of the transformation may be accounted for by $\hat{G}(k,t)\to\overline{G}(k,t)\equiv b(k,t)\hat{G}(k,t)$ for some factor $b(k,t)$. Furthermore, direct calculations from Eqs. (16)–(18) show that for both short and asymptotic times, $b(k,t)$ reduces to the exact form it must have,

$$b(k,t)=e^{-(1/2)k^2u_\star^2t^2}, \tag{20}$$

Equation (20) should be observed for all times [3]. However, for intermediate times, the MDIA predicts (very) small departure of $b(k,t)$ from the Gaussian factor (20).

Consider now the asymptotic solution of Eq. (16). Since $\hat{G}(k,t)$ becomes more and more peaked around $k=0$ as time goes on, the long time solution of Eq. (16) mainly depends [via the factor $pK(k,p,t)$] on the leading term of the series expansion in $k=0$ of the energy spectrum. Without losing much in generality, we shall take

$$E(k,t)\simeq v_\star^2\lambda_\star(\lambda_\star k)^\alpha\theta(t/\tau_\star),\quad \lambda_\star k\ll 1, \tag{21}$$

for some function $\theta(t/\tau_\star)$. Strictly speaking, we should restrict to the range $-1<\alpha$ in order to ensure the k integrability of $E(k,t)$; however, owing to the normalization (3) of $E(k,t)$, the limit $\alpha=-1$ of Eq. (21) can be interpreted as corresponding to a pure sweeping field for which $E(k,t) = \frac{3}{2} v_\star^2 \delta(k) \theta(t/\tau_\star)$. In Eq. (21), $\lambda_\star$ and $\tau_\star$ have, respectively, the dimensions of a length and of a time, but these quantities should not be confused with the characteristic length scale $l_\star$ and time scale $t_\star$ introduced in Eq. (4). Specifically, $\lambda_\star$ may only be identified with $l_\star$ if the corresponding integral in Eq. (4) is finite; that is, if $1<\alpha$ in Eq. (21). In contrast, $l_\star = \infty$ if $-1 \leqslant \alpha \leqslant 1$. Similarly $\tau_\star$ may only be identified with $t_\star$ if $\theta(t/\tau_\star)$ is integrable.

Using Eq. (21) in Eqs. (16)–(18), one may show that $\hat{G}(k,t)$ is asymptotically self-similar on all the k range where this function differs significantly from zero, in the sense that for $t \gg \min\{\lambda_\star/v_\star, \tau_\star\}$ it scales like

$$\hat{G}(k,t) \sim \hat{f}[k\sqrt{\langle \zeta^2(t)\rangle}],$$

where $\zeta(t)$ is the particle displacement at time t measured in any particular direction. The explicit forms of $\langle \zeta^2(t)\rangle$ and $\hat{f}(x)$ depend on the spectrum. Provided only that either $t_\star$ or $l_\star$ are finite, they are

$$\langle \zeta^2(t)\rangle \sim 2D_\star t, \quad \hat{f}(x) = \mathrm{e}^{-x^2/2}. \tag{22}$$

Thus, the process is Gaussian and diffusive as $t \to \infty$, and $G(r,t)$ satisfies a standard diffusion equation. The explicit value of $D_\star$ depends on $E(k,t)$ and D_0, but if $D_0 \ll D_\star$, its scales as $D_\star \propto v_\star^2 t_\star$ if $t_\star$ at least is finite and $t_\star \ll \min\{l_\star, \lambda_\star\}/v_\star$, and as $D_\star \propto v_\star l_\star$ if $l_\star$ at least is finite and $l_\star \ll v_\star \min\{t_\star, \tau_\star\}$. This second scaling is missed by the QNA and the QLA, but is predicted by the DIA. Indeed, Eq. (22) implies that the motion is asymptotically Markovian and the MDIA and the DIA are thus equivalent as $t \to \infty$. Note also that the finiteness of $l_\star$ may be proven necessary and sufficient for standard diffusion if $E(k,t) = E(k)$ (i.e., $t_\star = \infty$) and $D_0 > 0$ [5].

On the other hand, if both $l_\star = \infty$ and $t_\star = \infty$, we obtain the superdiffusive scalings

$$\begin{aligned} &\langle \zeta^2(t)\rangle \sim \tilde{c} v_\star \lambda_\star t \sqrt{\ln(v_\star t/\lambda_\star)} \quad (\alpha = 1), \\ &\langle \zeta^2(t)\rangle \sim \tilde{c} \lambda_\star^2 (v_\star t/\lambda_\star)^{4/(3+\alpha)} \quad (-1 \leqslant \alpha < 1), \end{aligned} \tag{23}$$

where $\tilde{c} > 0$ is a numerical constant. The expression for $\hat{f}(x)$ is rather cumbersome in this case, but we may prove that this function tends towards the Gaussian $\mathrm{e}^{-x^2/2}$ in both the limits $\alpha = 1$ and $\alpha = -1$ [the latter limit is exact; we recover Eq. (19) for the pure sweeping flow]. Furthermore, the departures from Gaussianity are rather weak in all the range $-1 < \alpha < 1$, with a maximum around $\alpha = 0$, and may be quantified by the flatness factor, which is estimated as

$$\lim_{t\to\infty} \frac{\langle \zeta^4(t)\rangle}{3\langle \zeta^2(t)\rangle^2} = 1 - \frac{1-\alpha^2}{5(5-\alpha)} \leqslant 1. \tag{24}$$

Thus, the distribution $G(r,t)$ has less weight in the tail beyond its mean than a Gaussian distribution, an effect presumably related to some amounts of trapping of the particle. Higher moments can be estimated as well, and suggest that $G(r,t)$ is approximately Gaussian for r of the order of $\langle \zeta^2(t)\rangle$ but has the shape of a quenched exponential for $r \gg \langle \zeta^2(t)\rangle$ (we were unable to give a rigorous proof of this point, however). Note also that in the present range, the MDIA is not asymptotically equivalent to the DIA since the dynamics is not Markovian as $t \to \infty$. Of course, since the sweeping is never negligible here, the DIA is out of its range of validity [6]. Specifically, even though the DIA leads to the scalings (23) for the particle mean square displacement [4], it overestimates the departure from Gaussianity in all the range $-1 \leqslant \alpha < 1$, and, in particular, it fails to predict the relaxation towards Gaussianity in the limit $\alpha = 1$.

Since the MDIA is exact for short times and gives reasonable results for long times, it leads to the confidence that, in contrast with the DIA, it is valid for all times. This conclusion can be confirmed by numerical integration of Eq. (16). For instance, for flows leading to standard diffusion, Eq. (16) reproduces for all times the results observed in the numerical simulations of particles motion in random flows [4], and, in particular, it accounts for the observed departures from Gaussianity at intermediate times (with a flatness factor smaller than 1).

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